PMM U.S.S.R.,Vol.53,No.6,pp.762-767,1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

A COMPARISON OF THE MODELS OF A THIN AND A COMPLETE VISCOUS SHOCK LAYER IN THE PROBLEM OF THE SUPERSONIC FLOW OF A VISCOUS GAS PAST BLUNT CONES*

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The flow of a viscous heat conducting supersonic gas past spherically blunted cones is used to compare the solutions of the equations of a thin (hypersonic) viscous shock layer (TVSL) with a given form of the shock wave (SW), with the solutions of the complete equations of a viscous shock layer (CVSL) in which the assumption that the shock layer is thin is not made and, which is important, the form of the SW is determined in the course of solving the problem. It is shown that a "successful" description of the form of the SW in solving the problem of hypersonic flow past a blunt cone within the framework of the equations of a TVSL provides, firstly, the possibility of obtaining the solution at considerable distances downstream and secondly, of sharpening the solution considerably, assuming that it can be obtained at all within the framework of the equations of the TVSL, compared with the commonly used asymptotic approach in which the form of the SW is assumed, in solving these equations, to be an equidistant form of the body.

The description of the supersonic flow of a viscous, heat conducting gas past a body using the simplified (parabolized) Navier-Stokes equations employs, as a rule, the Cheng two-layer model /1, 2/. According to Cheng, the whole perturbed region of the gas in front of the body can be divided into a region of viscous shock layer, and a transitional region corresponding to a density jump. The transition region is described by a system of ordinary differential equations which transforms, after appropriate simplifications, into the generalized Rankine-Hugoniot conditions. The region of the viscous shock layer is described using various systems of equations which are obtained either by an asymptotic method involving the separation of one or several small parameters of the problem (see e.g., /3/), or by a heuristic method involving an estimation of the contribution of each term of the system of equations /4/. In both cases the domain of applicability of the model is not clear in advance, and can be determined only be comparison with the complete Navier-Stokes equations. It should be noted that the domain of applicability of the simplified Navier-Stokes equations used to obtain a number of basic aerodynamic and thermal characteristics is found, as a rule, to be much wider than that of the formal asymptotic estimates.

1. Model of a thin viscous shock layer (TVSL). Historically, the first system of simplified Navier-Stokes equations, which is also the one most often used by virtue of its mathematical simplicity, is the system of equations of the TVSL (see e.g., /1-6/), obtained under the assumption that $\gamma \to 1$, $M_\infty \to \infty$, $\mathrm{Re}_\infty \to \infty$ (γ is the adiabatic ratio, M is the Mach number and Re is the Reynolds number). The equations of the TVSL contain all terms appearing in the equations of a non-viscous hypersonic shock layer /7/. The system of two-dimensional equations of the TVSL in a curvilinear system of coordinates attached to the body, has the form

$$\frac{\partial}{\partial x}(r^\nu \rho u) + \frac{\partial}{\partial y}(H_1 r^\nu \rho v) = 0 \tag{1.1}$$
$$\rho\left(\frac{u}{H_1}\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{uv}{RH_1}\right) = -\frac{1}{H_1}\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$
$$\omega\rho\frac{u^2}{RH_1} = \frac{\partial P}{\partial y}$$
$$\rho\left(\frac{u}{H_1}\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\mu}{\sigma}\left(\frac{\partial H}{\partial y} + \frac{\sigma-1}{2}\frac{\partial u^2}{\partial y}\right)\right)$$

Here x is the arc length of the contour of the body, y is the distance between the normal and the surface of the body, u and v are the physical velocity components in the x and y directions, H is the total specific enthalpy, σ is the Prandtl number, $R(x)$ is the radius of curvature of the contour of the body, H_1 and r are the metric Lamé coefficients, $H_1 = 1 + y/Q$, r is the distance between the given point of space and the axis of the body, $\nu = 0$

*Prikl.Matem.Mekhan.,53,6,963-969,1989

in the plane case and $\nu = 1$ in the axisymmetric case, and ω is a parameter describing the contribution of the centrifugal force $(0 \leqslant \omega \leqslant 1)$ (see Sect.3 for more details). System (1.1) must be supplemented by the equation of state.

We will use the condition of adhesion as the boundary condition on the body for the velocity, and we will assume that the surface of the body is either impermeable, or injection of prescribed intensity taken place from the surface of the body. The temperature at the wall is specified by one of the possible boundary conditions of the first, second or third kind. The terms describing the effect of slippage and temperature jump, under the assumption that the model of the TVSL is valid, are regarded as extraneous.

The generalized Rankine-Hugoniot conditions serve as boundary conditions on the inner (conditional) boundary of the SW for system (1.1), and are written in the hypersonic approximation as

$$\begin{gathered} v_s = u_s \operatorname{tg} \beta_s - k_s \sin \beta / \cos \beta_s \\ u_s = \cos \beta \cos \beta_s + k_s \sin \beta \sin \beta_s - (\mu \partial u / \partial y)_s / \sin \beta \\ H_s = 1 - (\mu_s / (\sigma \sin \beta))(\partial H / \partial y)_s \\ P_s = (1 - k_s) \sin^2 \beta + 1 / (\gamma M_\infty^2) \end{gathered} \tag{1.2}$$

Here $k_s = \rho_s^{-1}$, β_s and β is the angle of inclination of the SW to the body and to the axis of the body respectively.

The formulation of the problem is "supplemented" (the meaning of this expression will become clear below) by spcifying a priori the form of the leading SW which is, as a rule, equidistant from the surface of the body.

The sytem of equations of the TVSL is practically identical with the system of Prandtl equations. Its solution is even simpler than the solution of equations of the boundary layer, since here the problem connected with specifying the external pressure field does not arise (the field is determined in the course of solving the problem, from the third equation of system (1.1)). It should be noted that the problem is overdefined in the above formulation, since the SW serves as a free boundary whose position should be determined in the course of solving the problem. As was shown in /8/, this also follows from the results of /9/. The problem of determining the position of the SW is elliptic in the sense that a given point of the SW can also be influenced by a region situated further downstream even if the mechanism of transfer to weak perturbations in the upstream direction is not described by the system of equations of motion itself, as in the case of the TVSL (see Sect.2).

In order to solve this problem, it is necessary to use, in the course of determining the position of the SW, one or another iteration method, and this gives rise to well-known difficulties. Therefore, in the overwhelming majority of investigations using the equations of the TVSL, the position of the SW was specified, as a rule, as being equidistant from the surface of the body, and even without further revision.

2. ***Model of a complete viscous shock layer (CVSL).*** Further development of the model of the TVSL is represented by the system of equations of the CVSL proposed in /10/, and a similar system of parabolized Navier-Stokes equations /11/. Using the coordinates x, y, we can write the system of equations of the CVSL as follows:

$$\begin{gathered} \frac{\partial}{\partial x}(r^\nu \rho u) + \frac{\partial}{\partial y}(H_1 r^\nu \rho v) = 0 \\ \rho\left(\frac{u}{H_1}\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{uv}{RH_1}\right) = -\frac{1}{H_1}\frac{\partial P}{\partial x} + \\ \frac{1}{H_1^2 r^\nu}\frac{\partial}{\partial y}\left[H_1^2 r^\nu \mu\left(\frac{\partial u}{\partial y} - \frac{u}{H_1 R}\right)\right] \\ \rho\left(\frac{u}{H_1}\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \frac{u^2}{RH_1}\right) = -\frac{\partial P}{\partial y} \\ \rho\left(\frac{u}{H_1}\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}\right) = \frac{1}{H_1 r^\nu}\frac{\partial}{\partial y}\left(H_1 r^\nu \frac{\mu}{\sigma} \times \right. \\ \left.\left(\frac{\partial H}{\partial y} + \frac{\sigma - 1}{2}\frac{\partial u^2}{\partial y} - \frac{\sigma u^2}{RH_1}\right)\right) \end{gathered} \tag{2.1}$$

In the course of formulating the boundary conditions on the body for the system of equations of the CVSL, when the Reynolds numbers $(10 \lesssim \mathrm{Re}_\infty \lesssim 10^3)$ are moderate or small, we must take into account the effects of slippage and a temperature jump.

The boundary conditions on the SW, whose position is determined in the course of solving the problem, are represented for the equations of the CVSL by the generalized Rankine-Hugoniot conditions, which differ from (1.2) in the fact that the quantity $(\partial u/\partial u)_s$ in the second equation is replaced by $(\partial u/\partial y - u/(RH_1))_s$.

The main difference between systems (2.1) and (1.1) is, that the normal component of the velocity v is no longer assumed to be small, i.e. the condition $\gamma \to 1$ becomes optional.

Therefore the momentum equation projected on the normal to the body will contain on its left-hand side all acceleration terms. The system of Eqs.(2.1) contains all terms of the complete Navier-Stokes which contribute to the second boundary layer approximation in the small parameter $Re^{-1/2}$ for the inner as well as for the outer expansion. Using this approach we take into account the terms of order $O(1)$ and $O(Re^{-1/2})$, and neglect terms of order $O(Re^{-1})$ responsible for the molecular transfer of mass, momentum and energy along the body. Unlike the system of Eqs.(1.1), system (2.1) already describes the propagation of perturbations upstream in subsonic regions of the flow, and system (2.1) in these regions is elliptic. Since we always have the subonsic regions when the gas adheres to the walls, it follows that the Cauchy problem in x for the system of equations of the CVSL is ill-posed along the whole surface of the streamlined body.

Because of that, the system of equations of the CVSL is often solved using the method of establishment. It was remarked in /12/ that the time needed to solve the system of stationary equations of the CVSL on a digital computer using the method of establishment was comparable with the time needed to solve the system of complete Navier-Stokes equations. A method of solving the equations of the CVSL on a digital computer was given in /13/*. (*See also Utyuzhnikov S.V. A numerical method of solving the complete equations of a viscous shock layer, Dissertation, MFTI, Moscow, 1986.) The method was highly economical in memory and time, based on carrying out the global iterations, and allowed a tenfold reduction in computer time. Nevertheless, solving the system of equations of the CVSL remains a fairly complicated and time-consuming process. Therefore, whenever the accuracy requirements allow, it is better to use simpler models such as the model of the TVSL, although in this case the amount of irremovable error inherent in the model must be clearly described.

The error in the model of the TVSL in the case of smooth spheres and a hyperboloid or paraboloid of revolution was studied earlier (e.g. in /14/), where it was proposed not to fix a priori the form of the SW in the course of solving the equations of the TVSL, but to refine it with the help of global iterations using the integral relation of the mass balance of the gas /13/. Flow past a hyperboloid of revolution with an aperture half-angle of 10° was solved as an example to illustrate that the latter method considerably reduced was error of the model within the framework of the equations of the TVSL.

3. *Comparison of the models. Results of computations.* The error of the model of the TVSL was investigated for the case of flow past a spherically blunted cone, by comparing the solutions of the system of equations of the thin, and the complete viscous shock layer. The equations of the TVSL were solved here with the position of the SW more acurately defined, as in /4/. In studying the error of the model of the TVSL the authors used existing information which states that the difference between the solutions of the system of equations of the CVSL and the system of complete Navier-Stokes equations for the distribution of the pressure, friction and heat fluxes does not exceed, in the case of flows past blunt cones, 1-2% for $Re_\infty > 10^3$. The system of equations of the CVSL was solved by numerical methods /13/.

In carrying out the numerical computations, we used independent variables of the Dorodnitsyn type ξ, η /4/

$$\xi = x, \quad \eta(x, y) = \int_0^y \rho^{-\nu}\Delta^{-1} dn, \quad \Delta = \int_0^{y_s} \rho^{-\nu} dn \tag{3.1}$$

where $y_s(x)$ is the separation of the SW from the body.

The difference scheme used had second order of approximation in the derivatives in x, and fourth order in y. In order to solve motion with large gradients, we used a variable distribution of the steps of the difference mesh in η, which were chosen at every point depending on the variation of the function in its neighbourhood. The nodes of the difference mesh in ξ were distributed in such a manner that one of the nodes was at the point of conjugation of the sphere with the cone, representing the point of discontinuity in the curvature of the generatrix. When solving the equations of the CVSL, we determined the point of conjugation with the second-order approximation using the exact relations for the discontinuity in the first-order and second-order derivatives of the functions sought, in the intrinsic coordinate system obtained in /15/.

When the model of the TVSL was used, we specified the position of the SW on the spherical part using the following approximate formula /9/:

$$\operatorname{tg}\beta_s = c\left[\left((2c-1)c^{-2} + \operatorname{tg}^2\alpha\right)^{1/2} - \operatorname{tg}\alpha\right] \tag{3.2}$$
$$c = \tfrac{1}{2}R_s(0)\left(R(0) + y_s(0)\right)^{-1}$$

where $R_s(x)$ is the radius of curvature of the SW. In estimating $y_s(0)$ and $R_s(0)$, we used the well-known approximation formulas /16/, and the following semi-empirical formula was used for the conical part of the body (see the paper to Utyuzhnikov cited in the earlier footnote):

$$\operatorname{tg}\beta_s = \operatorname{tg}\beta_s(x_c)\exp[k_c(x - x_c)]$$
$$k_c = \frac{1}{H_{1s}\operatorname{tg}\beta_s}\frac{d\operatorname{tg}\beta_s}{dx}\bigg|_{x_c-0} - \frac{1}{H_{1s}\sin\beta_s\cos\beta_s R}\bigg|_{x_c-0} \tag{3.3}$$

(x_c is the coordinate of the point of conjugation). Comparison with the numerical solutions shows that the composite formula (3.2), (3.3) gives a fairly good approximation, the error being of the order of 10-30%. The results of the computations given below correspond to the flows of a real gas past spherically blunted cones for $M_\infty = 20$, $\mathrm{Re}_\infty = 10^4$, $T_w = 0.5$ (T_w is a temperature factor).

Fig.1 shows the distribution along the surface of the cone with an aperture half-angle of 30°, of the heat flux in the form of the Stanton number $\mathrm{St} = \lambda(\partial T/\partial y)_w/[\rho_\infty v_\infty H_\infty(1 - T_w)]$, and of the pressure referred to $\rho_\infty v_\infty^2$, z is the coordinate along the cone axis measured from the stagnation point. The dashed lines correspond to computations based on the equations of the TVSL, with prescribed position of the SW (3.2), (3.3). The curve shown by light dots corresponds to a computation based on the model of the TVSL where the position of the SW is determined using iterative methods, and the curve shown by dark dots refers to the computation based on the model of the CVSL. We see from the graphs that, unlike the class of smooth bodies, the iterative refinement of the position of the SW based on the model of the TVSL, yields approximately the same error in determining the values of the functions sought on the body, as when the approximate formulas (3.2), (3.3) are used. The difference in the values of the local coefficient of friction $c_f = {}^1/_2\mu_w(\partial u/\partial y)_w/(\rho_\infty v_\infty^2)$ obtained with help of the model of the TVSL was 10-15%.

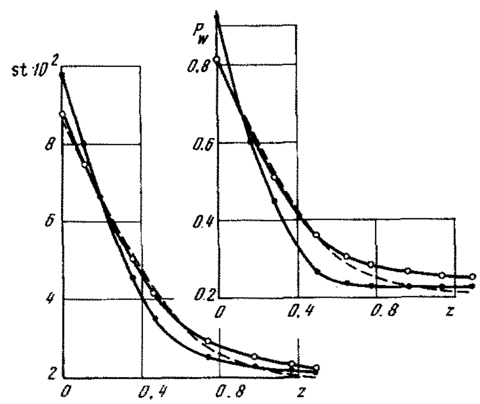

Fig.1

The model of the TVSL appears to be completely acceptable for studying flows past cones with an aperture half-angle exceeding 45°, since the admissible error allowed in this case does not exceed 10%. Fig.2 shows, as an example, the computed data on the distribution of the values of the local coefficient of friction c_f along the surface of the cone with an aperture half-angle of 45°. In the case of flows part thin blunt cones the model of the TVSL is either inapplicable, or partly applicable. (The results of computing the distribution of pressure along the surface of the cone with an aperture half-angle of 15° are given in Fig.2; the notation used is the same as in Fig.1).

We note that in computing the data we assumed that the value of the parameter ω in (1.1) was zero, since the approximation used for this class of bodies is more accurate /16/. Moreover, a value of ω different from zero leads to a "non-physical" jump in the presure at the point of conjugation of the sphere with the cone.

For each model of the flow, we compared the accuracy with which the integral laws of conservation, based on the complete system of Navier-Stokes equations, were satisfied. The control volume was determined by rotating the generatrix ABCDE (Fig.3) about the axis of the body, where the line BC is parallel to the normal to the surface of the body, the line DC is parallel to the axis of the body and the point C lies on the outer boundary of the SW. The relative difference in the incoming mass (subscript +) and outgoing mass (subscript -), momentum and energy fluxes, was calculated as a percentage using the relation

$$\delta Q = |(Q_+ + Q_-)/Q_-| \times 100 \tag{3.4}$$

Here, we mean by the incoming flow the flow through the surface *AB*, and by the outgoing flow we mean the flow through the surface *CD*, as well as the momentum and energy losses at the surface *DE*, i.e., on the bodies.

The table shows some versions of the comparisons made. Here δQ_m, δQ_i, δQ_e are the disbalances in the mass, momentum and energy fluxes respectively, calculated using formula (3.4). TVSL(GI) denotes the use of the TVSL model in which the position of the SW is determined by iteration. Compuations were carried out for various aperture half-angles θ of the cone. In the case of the model of the TVSL $\delta Q_m = 7$ at $\theta = 45°$ and $\delta Q_m = 5$ at $\theta = 30°$. Since in the global iteration the position of the SW is determined from the condition of mass balance, it follows that in the CVSL and TVSL(GI) versions the mass disbalance is zero. When $\theta = 15°$, the computation of the TVSL used as the starting point the computation carried out earlier for $\theta = 30°$ and we have therefore no version of the TVSL for $\theta = 15°$ in the table.

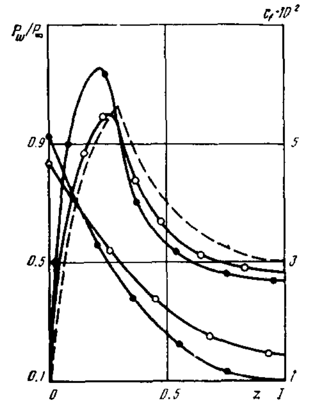

Fig.2

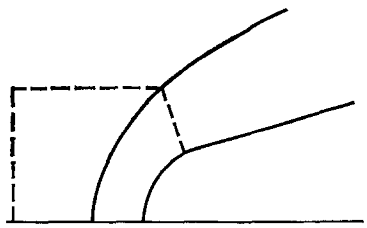

Fig.3

Using the results of the present paper and of /14/, we can draw the following conclusions. The model of the TVSL yields acceptable results, when the position of the SW is specified with sufficient accuracy, in the case of flows past smooth blunt hyperboloids or paraboloids of revolution. The traditional assignment of the form of the SW as being equidistant from the contour of the body can result in appreciable error, especially in determining the pressure on the body, and this considerably narrows the domain of applicability of this model.

Model	$\theta = 45°$		30°		15°	
	δQ_i	$\delta Q_e\cdot10^2$	δQ_i	$\delta Q_e\cdot10^2$	δQ_i	$\delta Q_e\cdot10^2$
CVSL	0.4	2	0.3	2	0.5	3
TVSL(Gi)	1.7	0.4	5.6	0.5	10.6	2
TVSL	1.6	0,2	5.7	1		

When it comes to numerical modelling of the flow past a blunt cone, the model of the TVSL can be used in practice only in the case when the aperture half-angles of the cone are large ($\geqslant 40°$). Also, specifying the form of the SW as equidistant from the body leads to a "non-physical" discontinuity appearing in the curvature of the SW and hence to a jump in the component of the pressure gradient tangential to the body. We recommend the use of more-accurate approximations (e.g. approximation (3.2), (3.3)) for specifying the position of the SW. The resulting error in the distribution of the pressure, friction and heat transfer over the body remains the same, on the whole, in the case of iterative refining of the position of the SW within the framework of the model of the TVSL.

REFERENCES

1. GERSHBEIN E.A., PEIGIN S.V. and TIRSKII G.A., Supersonic flows past bodies at small and moderate Reynolds numbers. Itogi Nauki i Tekhniki. Mekhanika zhidkosti i gaza. VINITI, 19, Moscow, 1985.
2. PEIGIN S.V. and TIRSKII G.A., Three-dimensional problems of super- and hypersonic flows of a viscous gas past bodies. Itogi Nauki i Tekhniki. Mekhanika zhidkosti i gaza. VINITI, 22, Moscow, 1988.
3. BUSH W.B., On the viscous hypersonic blunt body problem, J. Fluid Mech. 20, 3, 1964.

4. TIRSKII G.A., On the theory of hyersonic viscous, chemically reactive gas flow with injection, past plane and axisymmetric blunt bodies. Nauch. Tr. In-ta Mekhan., MGU, 39, 1975.
5. DAVIS R.T., RUBIN S.G., Non-Navier Stokes viscous flow computations, Comput. and Fluids. 8, 1, 1980.
6. GERSHEIN E.A., SHCHELIN V.S. and YUNITSKII S.A., Numerical and approximately analytic solutions of equations of three-dimensional hypersonic viscous shock layer at moderately small Reynolds numbers. Hypersonic Three-dimensional Flows in the Presence of Physical and Chemical Transformations. Izd-vo MGU, Moscow, 1981.
7. CHERNYI G.G., Gas Flow at High Supersonic Velocity. Fizmatgiz, Moscow, 1959.
8. SRIVASTAVA B.N., WERLE M.J. and DAVIS R.T., Numerical solutions of hypersonic viscous shock-layer equations. AIAA Journal. 17, 1, 1979.
9. TIRSKII G.A., Approximal analytical determination of departure of the leading SW in supersonic flows past blunt bodies. Mathematical modelling of non-stationary problems of mechanics of continua. Vsesoyus. Zaoch. Mashinostroit. In-t, Moscow, 1985.
10. DAVIS R.T., Flügge-Lotz I. Second - order boundary-layer effects in hypersonic flow past axisymmetric blunt bodies. J. Fluid Mech. 20, 4, 1964.
11. TOLSTYKH A.I., On the numerical solution of the problems of supersonic flows of viscous gas past blunt bodies. Zh. vychisl. Mat. mat. Fiz., 6, 1, 1966.
12. GOLOVACHEV YU.P. and POPOV F.D., Calculation of supersonic viscous gas flow past blunt bodies at large Reynolds numbers. Zh. vychisl. Mat. mat. Fiz. 12, 5, 1972.
13. VASIL'YEVSKII S.A., TIRSKII G.A. and UTYUZHNIKOV S.V., Numerical method of solving the equations of a viscous layer. Zh. vychisl. Mat. mat. Fiz., 27, 5, 1987.
14. VASIL'YEVSKII S.A. and TIRSKII G.A., Numerical method of solving the equations of a viscous shock layer. Studies of hypersonic aerodynamics and heat exchange taking physico-chemical non-equilibrium reactions into account. Iz-vo MGU, Moscow, 1987.
15. UTYUZHNIKOV S.V., The use of a curvilinear system of coordinates attached to the body in the numerical modelling of flows past blunt cones. Proceedings of the 10-th Conference of Young Scientists, MFTI, Dolgoprudnyi, 1985.
16. LUNEV V.V., Hypersonic Aerodynamics. Mashiostroyeniye, Moscow, 1975.

Translated by L.K.

PMM U.S.S.R.,Vol.53,No.6,pp.767-772,1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

THE EFFECT OF CONJUGATED AND RADIANT HEAT EXCHANGE ON THE PROCESS OF NON-STATIONARY COMBUSTION OF THE PRODUCTS OF INTENSE GASIFICATION OF A SOLID IN A STREAM OF GAS*

V.M. AGRANAT AND D.A. GUBIN

This paper develops further the results obtained in /1-4/ and uses the approximate mathematical model /2/ of the combustion of the products of intense gasification of the neighbourhood of the leading stagnation point of the body to analyse the effect of the conjugation parameters on the heat exchange, radiation and other factors on the conditions of uniqueness and stability of the stationary combustion modes. When the gasification is carried out at a constant mass flow rate, an analogy is established, depending on the relations between the parameters of the problem, between the model in question and the models of a homogeneous chemical continuous action reactor with a fluidized catalyst layer, and a reactor with a temperature regulator /5/. Simple necessary conditions for the instability of the stationary modes and the appearance of self-excited oscillations are obtained. A strong stabilizing influence of the conjugated heat exchange and intense injection on the combustion

Prikl.Matem.Mekhan.,53,6,970-975,1989